Radial Basis Function Networks

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A **radial basis function network (RBFN)** is a neural network with a graph $G = (U, C)$ that satisfies the following conditions

(i) $U_{\text{in}} \cap U_{\text{out}} = \emptyset$,

(ii) $C = (U_{\text{in}} \times U_{\text{hidden}}) \cup C', \quad C' \subseteq (U_{\text{hidden}} \times U_{\text{out}})$

The network input function of each hidden neuron is a **distance function** of the input vector and the weight vector, that is,

$$\forall u \in U_{\text{hidden}}: \qquad f_{\text{net}}^{(u)}(\vec{w}_u, \vec{\text{in}}_u) = d(\vec{w}_u, \vec{\text{in}}_u),$$

where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a function satisfying $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n :$

$$\begin{array}{rll} (i) & d(\vec{x}, \vec{y}) = 0 \quad \Leftrightarrow \quad \vec{x} = \vec{y}, & \\ (ii) & d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x}) & \text{(symmetry)}, \\ (iii) & d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) & \text{(triangle inequality)}. \end{array}$$

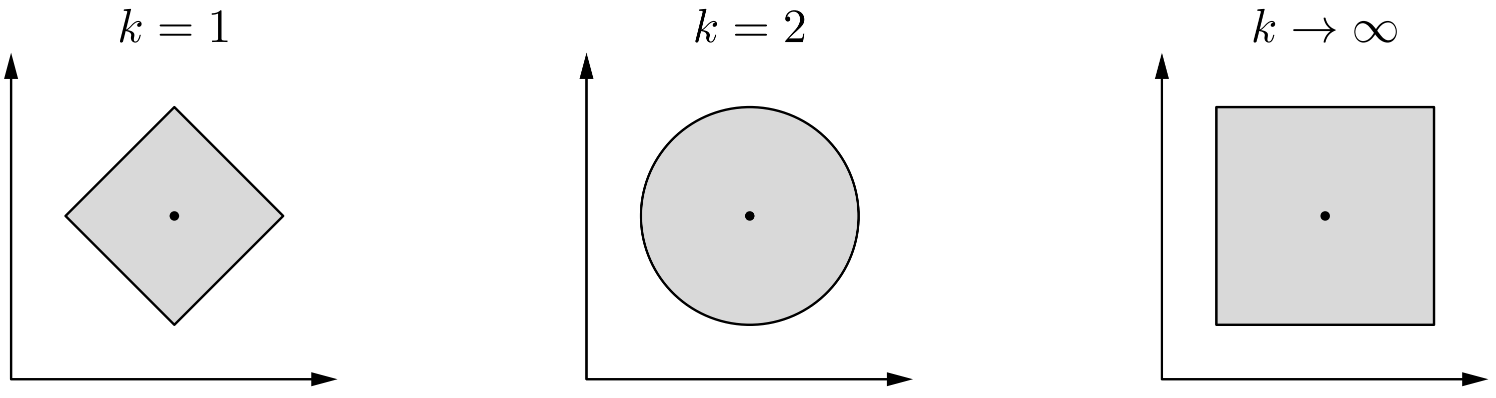
Distance Functions

Illustration of distance functions: Minkowski Family

$$d_k(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^k \right)^{\frac{1}{k}}$$

Well-known special cases from this family are:

$k = 1$:	Manhattan or city block distance,
$k = 2$:	Euclidean distance,
$k \to \infty$:	maximum distance, that is, $d_\infty(\vec{x}, \vec{y}) = \max_{i=1}^{n} |x_i - y_i|$.



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The network input function of the output neurons is the weighted sum of their inputs:

$$\forall u \in U_{\text{out}}: \qquad f_{\text{net}}^{(u)}(\vec{w}_u, \vec{\text{in}}_u) = \vec{w}_u^\top \vec{\text{in}}_u = \sum_{v \in \text{pred}\,(u)} w_{uv} \,\text{out}_v\,.$$

The activation function of each hidden neuron is a so-called **radial function**, that is, a monotonically decreasing function

$$f: \mathbb{R}_0^+ \to [0,1] \quad \text{with} \quad f(0) = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0.$$

The activation function of each output neuron is a linear function, namely

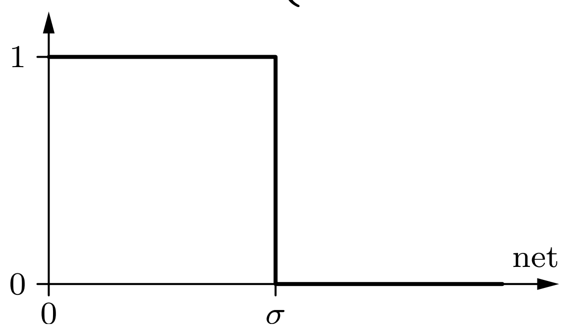
$$f_{\text{act}}^{(u)}(\text{net}_u, \theta_u) = \text{net}_u - \theta_u.$$

(The linear activation function is important for the initialization.)

Radial Activation Functions

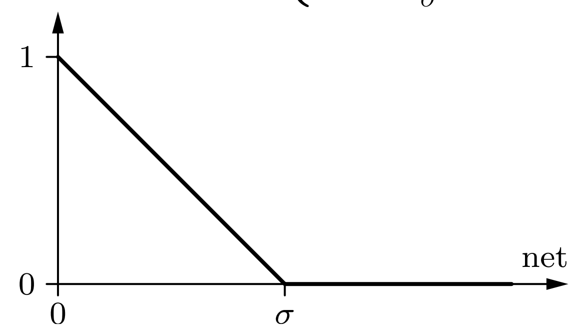
rectangle function:

$$f_{\text{act}}(\text{net}, \sigma) = \begin{cases} 0, & \text{if net} > \sigma, \\ 1, & \text{otherwise.} \end{cases}$$



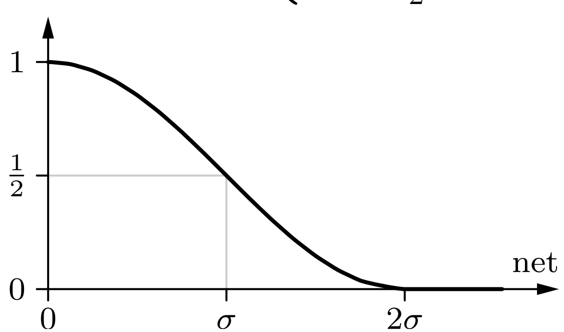
triangle function:

$$f_{\text{act}}(\text{net}, \sigma) = \begin{cases} 0, & \text{if net} > \sigma, \\ 1 - \frac{\text{net}}{\sigma}, & \text{otherwise.} \end{cases}$$



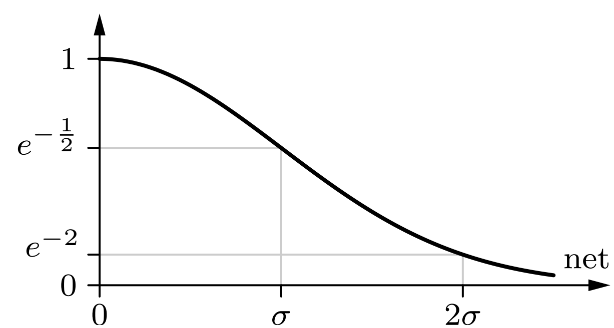
cosine until zero:

$$f_{\text{act}}(\text{net}, \sigma) = \begin{cases} 0, & \text{if net} > 2\sigma, \\ \frac{\cos\left(\frac{\pi}{2\sigma}\text{net}\right)+1}{2}, & \text{otherwise.} \end{cases}$$



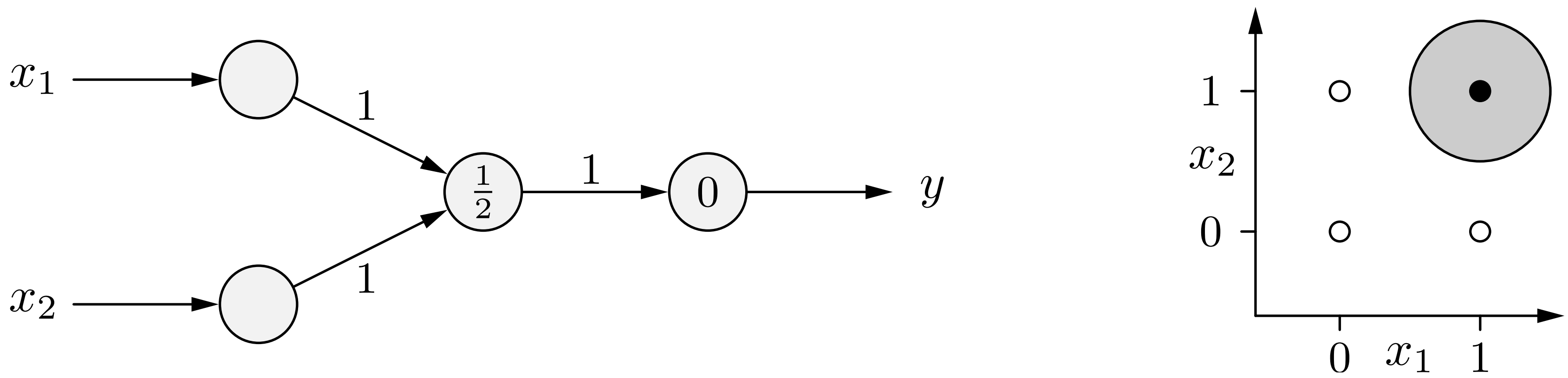
Gaussian function:

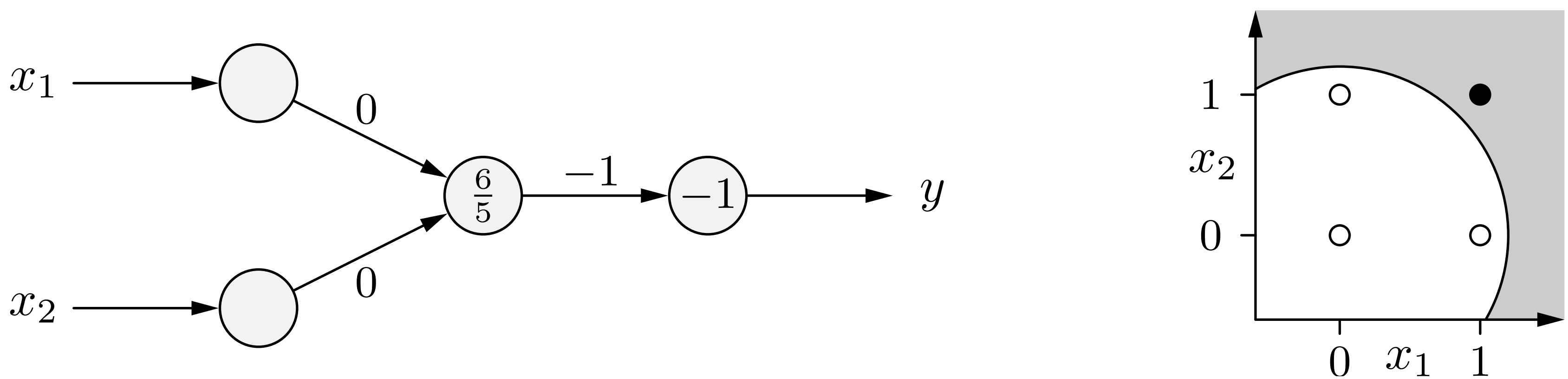
$$f_{\text{act}}(\text{net}, \sigma) = e^{-\frac{\text{net}^2}{2\sigma^2}}$$



Radial Basis Function Networks: Examples

Radial basis function networks for the conjunction $x_1 \wedge x_2$

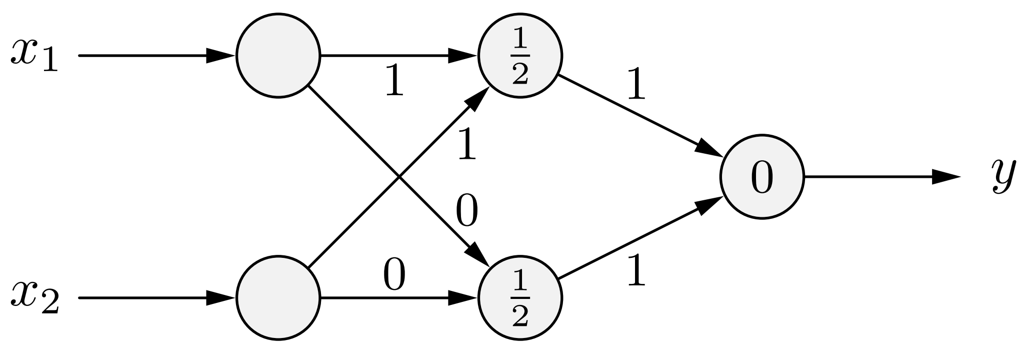


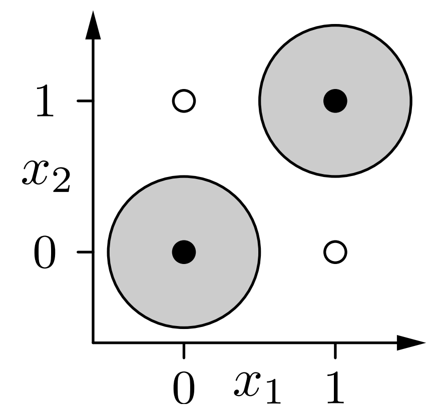


Radial basis function networks for the biimplication $x_1 \leftrightarrow x_2$

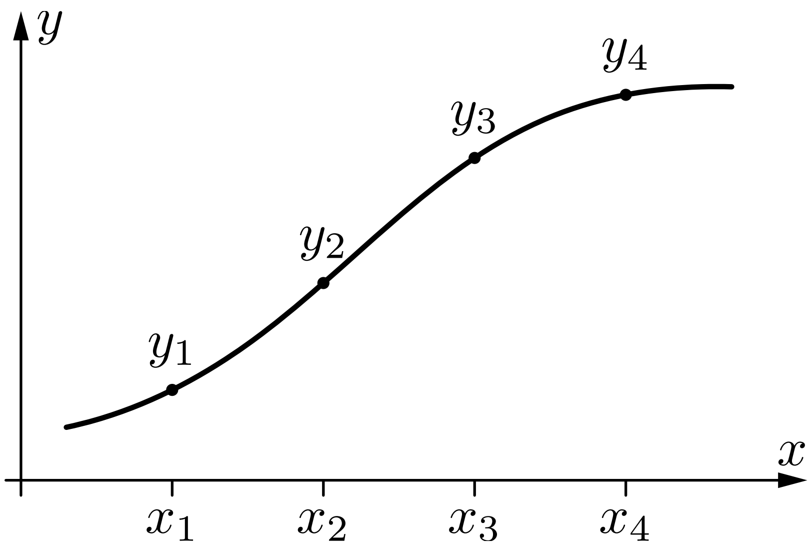
Idea: logical decomposition

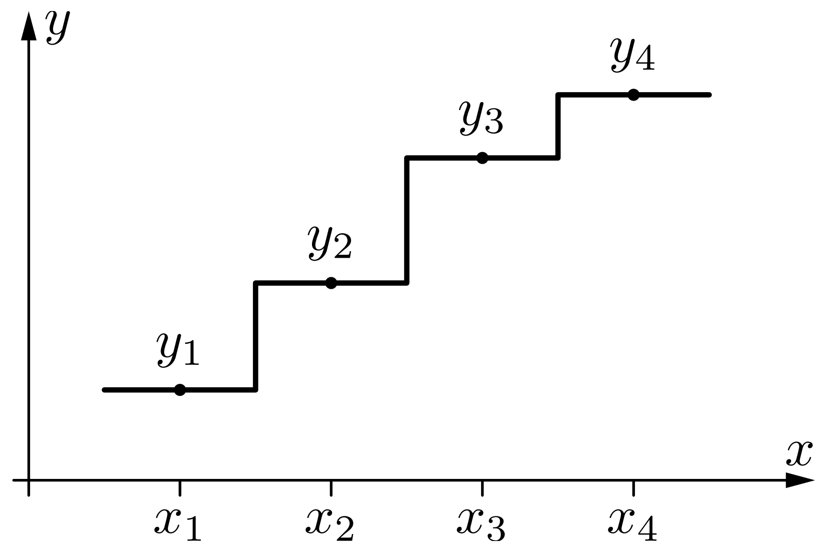
$$x_1 \leftrightarrow x_2 \quad \equiv \quad (x_1 \wedge x_2) \vee \neg(x_1 \vee x_2)$$



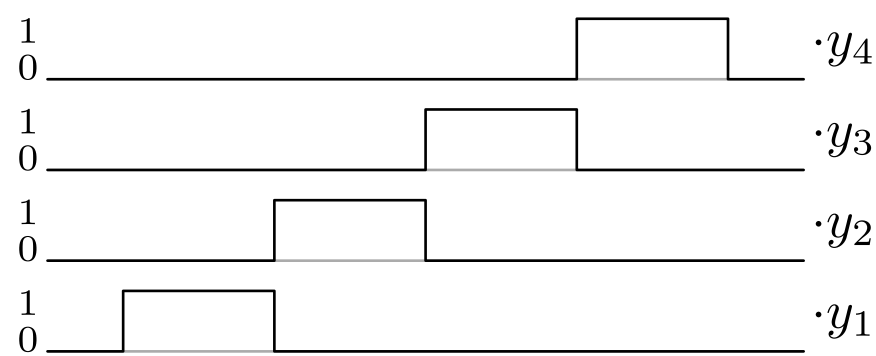


Radial Basis Function Networks: Function Approximation

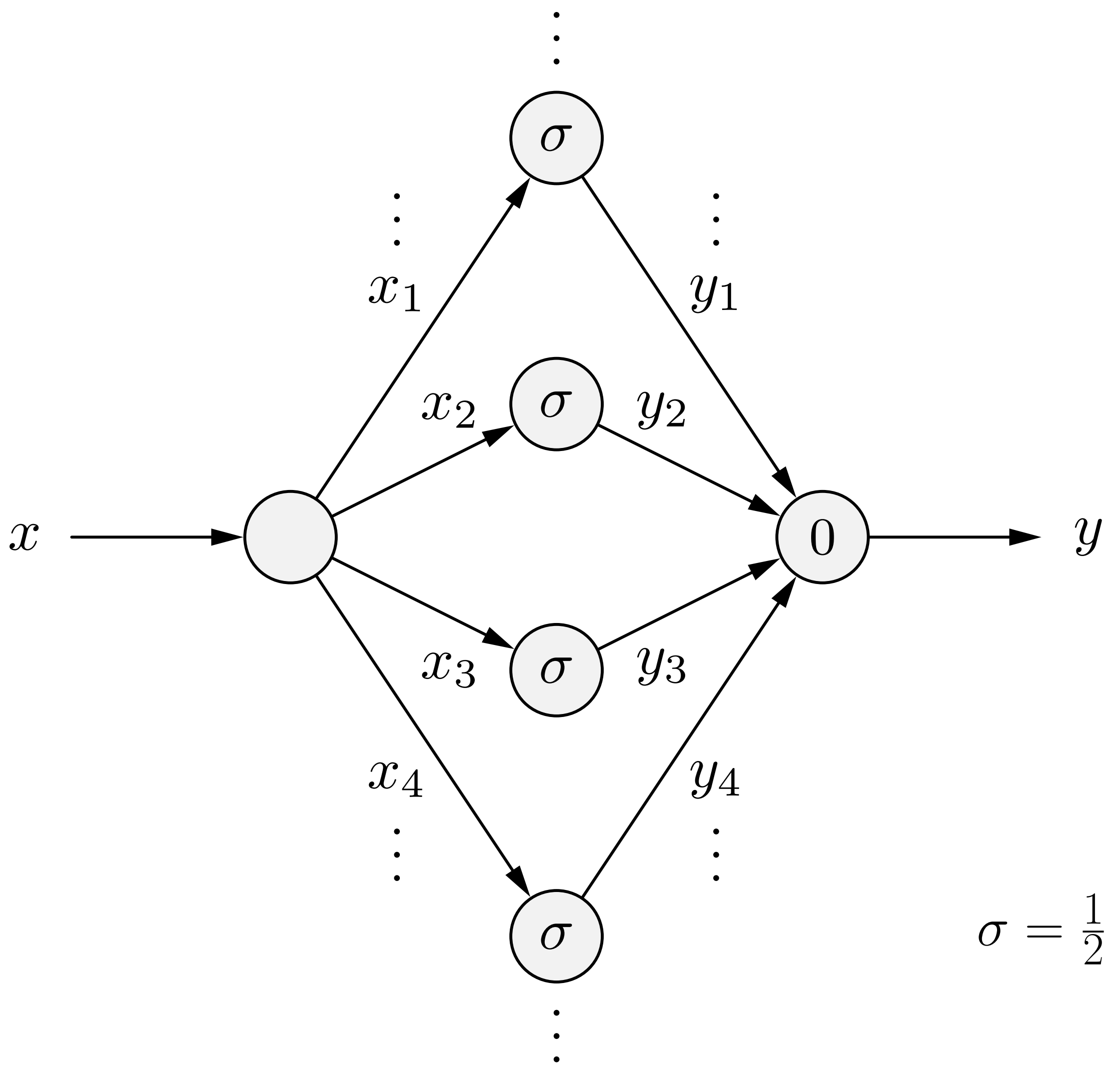




Approximation of a function by rectangular pulses, each of which can be represented by a neuron of an radial basis function network.



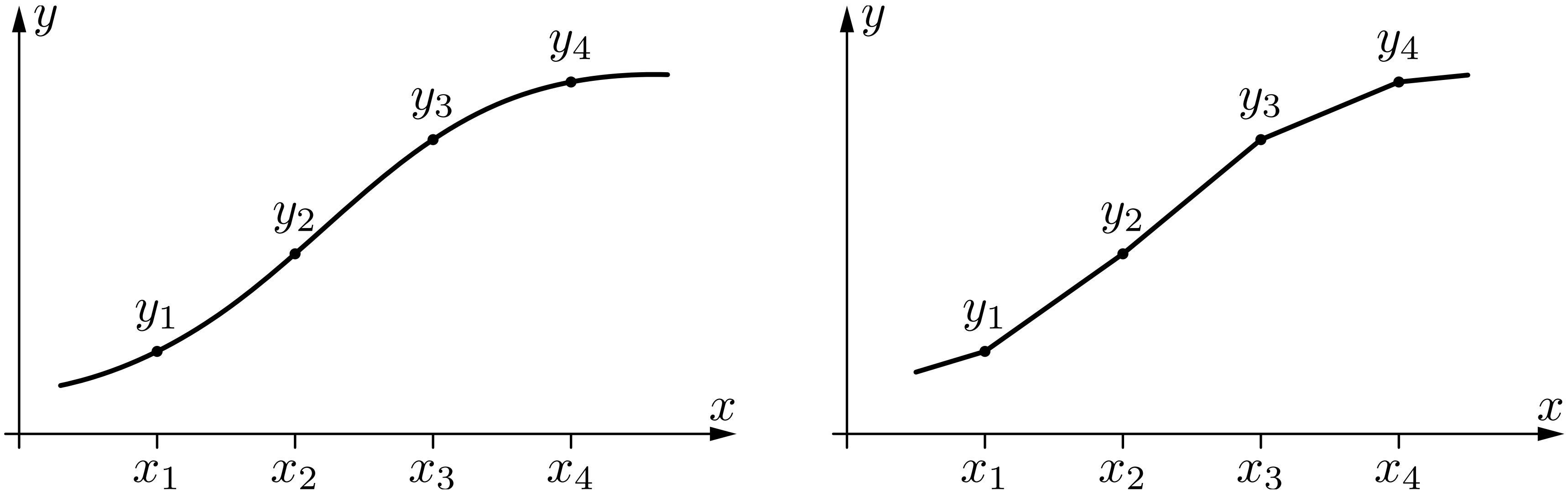
Radial Basis Function Networks: Function Approximation



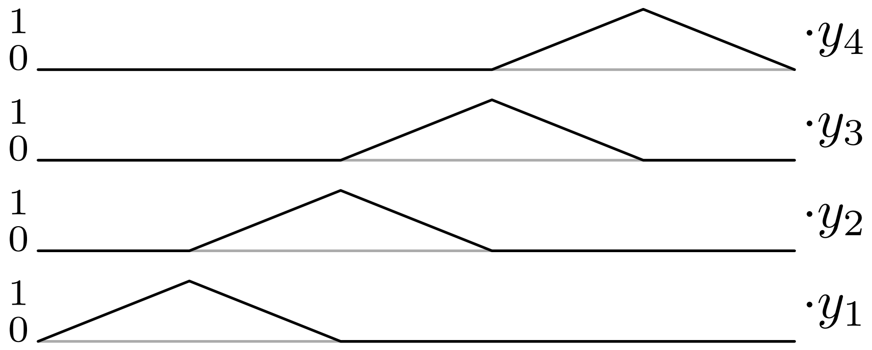
$$\sigma = \frac{1}{2}\Delta x = \frac{1}{2}(x_{i+1} - x_i)$$

A radial basis function network that computes the step function on the preceding slide and the piecewise linear function on the next slide (depends on activation function).

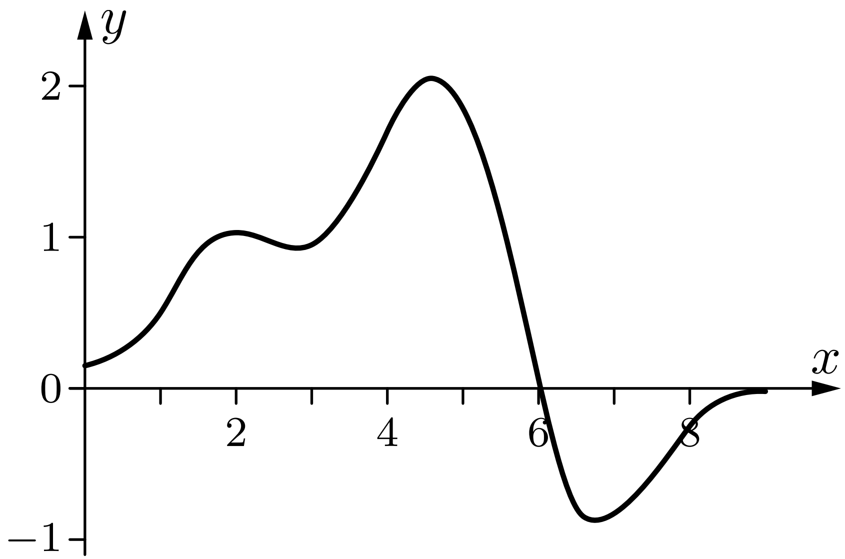
Radial Basis Function Networks: Function Approximation

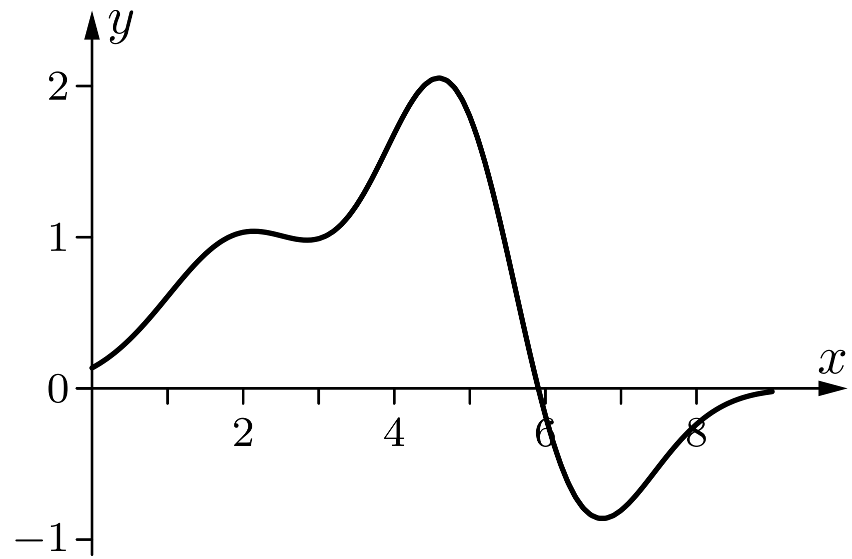


Approximation of a function by triangular pulses, each of which can be represented by a neuron of an radial basis function network.

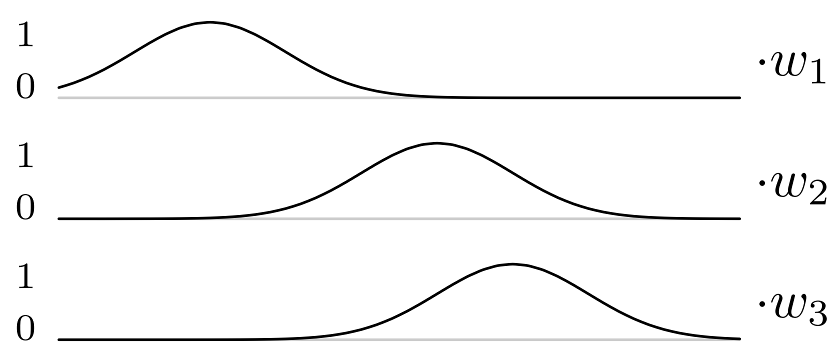


Radial Basis Function Networks: Function Approximation



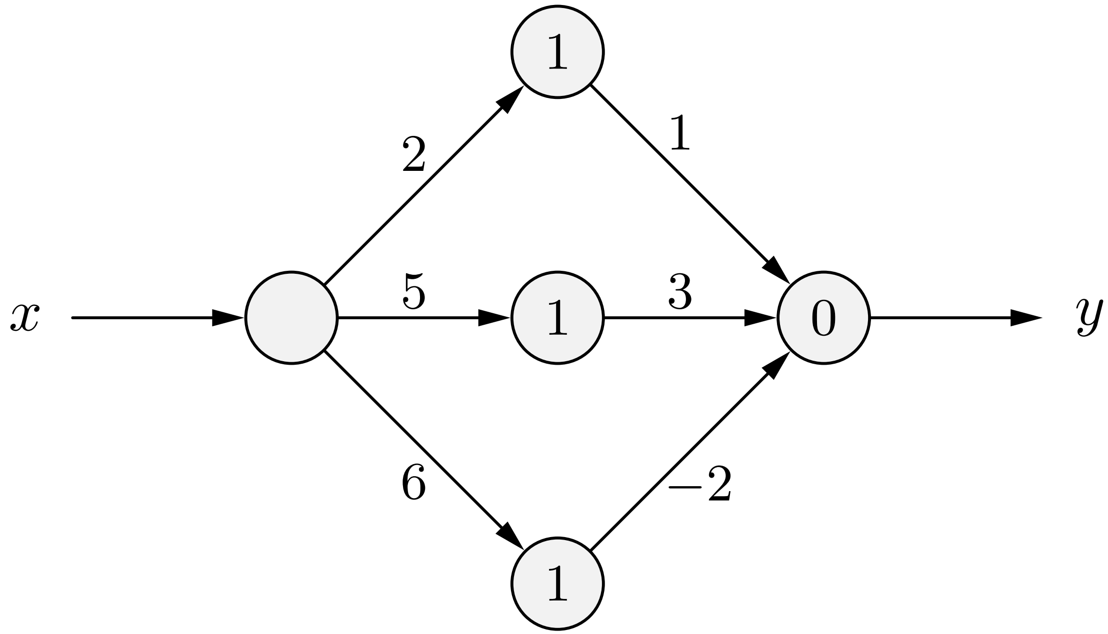


Approximation of a function by Gaussian functions with radius $\sigma = 1$. It is $w_1 = 1$, $w_2 = 3$ and $w_3 = -2$.



Radial Basis Function Networks: Function Approximation

Radial basis function network for a sum of three Gaussian functions



- The weights of the connections from the input neuron to the hidden neurons determine the locations of the Gaussian functions.
- The weights of the connections from the hidden neurons to the output neuron determine the height/direction (upward or downward) of the Gaussian functions.